Spectrum of turbulence with temperature gradient (in the atmosphere)

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Spectrum of turbulence with temperature gradient

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Abstract. The problem of calculating the energy spectra of velocity and temperature fluctuations in a turbulence which is maintained by a permanent, large scale temperature gradient is studied, using a repeated cascade theory. A pair of coupled spectral equations is derived containing a turbulent momentum transport coefficient, which is identified as the eddy viscosity. An equation is formed which, when solved, leads immediately to an expression for the eddy viscosity. Solutions for the velocity and temperature spectra are found in the production-transfer, inertial and dissipation subranges. The Kolmogorov spectrum is recovered in the inertial subrange. The spectra have the form of power laws with exponents ranging from -1 to -7. Some observed spectra illustrate the predicted behaviour. A comparison is made between the methods and results of this paper and the work of other researchers, notably McComb.

1. Introduction

Turbulence which occurs naturally must be driven by an energy source, which also normally causes the turbulence to be anisotropic. Geophysical turbulence is commonly caused by either wind shear or a temperature gradient (or both). In this paper, atmospheric turbulence which is driven by the latter mechanism is considered. The problem is complicated by the fact that both the fluctuating velocity and the temperature fields must be taken into account, leading to a coupled system of partial differential equations for the velocity and temperature spectra.

The problem will make use of the new repeated cascade theory of turbulence recently developed by Tchen (1973), who made an application principally to homogeneous, isotropic and incompressible turbulence. This theory has also been applied to wave propagation in a turbulent plasma by Martens and Jen (1975). Reference may be made to the above cited paper of Tchen for a detailed formulation of the repeated cascade theory; only the most relevant aspects will be outlined here.

In this theory, the fluctuating, or turbulent, part of each variable is represented as a series of components. For example, if $\boldsymbol{u}'$ is the turbulent constituent of the total velocity, $\boldsymbol{u}$, then $\boldsymbol{u}'$ is written as

$$\boldsymbol{u}'=\sum_{\alpha} \boldsymbol{u}^{(\alpha)} \tag{1.1}$$

where $\boldsymbol{u}^{(\alpha)}$ is called the component of rank α. Higher ranks are associated with increasing degrees of randomness in the sense that the correlation length, L_α, of

$\langle u_i^{(\alpha)}(\boldsymbol{x}, t)u_j^{(\alpha)}(\boldsymbol{x}+\boldsymbol{r}, t)\rangle/\langle(\boldsymbol{u}^{(\alpha)})^2\rangle$ decreases as α increases. Thus, if $\langle\ \ \rangle^{(\alpha)}$ denotes an average over a length scale L_α, then

$$\langle \boldsymbol{u}^{(\alpha)}\rangle^{(\beta)} = \begin{cases} 0, & \beta \leqslant \alpha \\ \boldsymbol{u}^{(\alpha)}, & \beta > \alpha. \end{cases} \tag{1.2a}$$

Consequently,

$$\boldsymbol{u}^{(\alpha)} = \langle \boldsymbol{u}' \rangle^{(\alpha+1)} - \langle \boldsymbol{u}' \rangle^{(\alpha)}$$

and the averaging operator $\langle\ \ \rangle^{(\alpha)}$ can be used to distinguish amongst the various ranks. It is important to note that the series (1.1) is not an expansion in terms of a small expansion parameter (which does not exist in turbulence).

Continuing to use the velocity as an example, the Fourier transform of $\boldsymbol{u}^{(\alpha)}(\boldsymbol{x}, t)$ is $\boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t)$, where

$$\boldsymbol{u}^{(\alpha)}(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{k}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}} \boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t).$$

From (1.2a), and treating $\langle\ \ \rangle^{(\alpha)}$ as the result of averaging many realizations of the turbulent field, it follows that

$$\langle \boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t)\rangle^{(\beta)} = \begin{cases} 0, & \beta \leqslant \alpha \\ \boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t), & \beta > \alpha \end{cases} \tag{1.2b}$$

and the Fourier transform $\boldsymbol{u}'(\boldsymbol{k}, t)$ of $\boldsymbol{u}'(\boldsymbol{x}, t)$ is

$$\boldsymbol{u}'(\boldsymbol{k}, t) = \sum_\alpha \boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t).$$

In view of (1.2b), the principal contribution to $\boldsymbol{u}'(\boldsymbol{k}, t)$ coming from $\boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t)$ will be confined to a distinct wavenumber range $k_{\alpha-1} < k < k_\alpha$ so that approximately

$$\boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t) \cong \begin{cases} \boldsymbol{u}'(\boldsymbol{k}, t), & k_{\alpha-1} < k < k_\alpha \\ 0, & k_\alpha < k < k_{\alpha-1} \end{cases} \tag{1.3}$$

where k_α and $k_{\alpha-1}$ are the upper and lower cutoff wavenumbers for the transform of rank α. A more rigorous derivation has been given by Tchen (1973).

The three-dimensional energy spectrum tensor $F_{ij}^{(\alpha)}(\boldsymbol{k}, t)$ corresponding to a spatial average $\langle\ \ \rangle^{(\alpha)}$ is defined by

$$\tfrac{1}{2}\langle u_i^{(\alpha)}(\boldsymbol{x}, t)u_j^{(\alpha)}(\boldsymbol{x}, t)\rangle^{(\alpha)} \equiv \int \mathrm{d}\boldsymbol{k}\, F_{ij}^{(\alpha)}(\boldsymbol{k}, t)$$

and is

$$F_{ij}^{(\alpha)}(\boldsymbol{k}, t) = \tfrac{1}{2}\chi_\alpha \langle u_i^{(\alpha)}(\boldsymbol{k}, t)u_j^{(\alpha)}(-\boldsymbol{k}, t)\rangle^{(\alpha)} \tag{1.4}$$

where $\chi_\alpha \equiv (2\pi/L_\alpha)^3$ arises from the averaging. In terms of the spectrum $F_{ij}(\boldsymbol{k}, t)$ corresponding to a spatial average of $(\boldsymbol{u}')^2$:

$$F_{ij}^{(\alpha)}(\boldsymbol{k}, t) = \begin{cases} F_{ij}(\boldsymbol{k}, t), & k_{\alpha-1} < k < k_\alpha \\ 0, & k_\alpha < k < k_{\alpha-1}. \end{cases} \tag{1.5}$$

The energy spectrum $F_{ij}^{(\alpha)}$ is related to quantities arising from the second order Lagrangian correlation

$$R_{ij}^{(\alpha)}(\tau)\equiv\langle u_i^{(\alpha)}(\boldsymbol{x}(t),t)u_j^{(\alpha)}(\boldsymbol{x}(t+\tau),t+\tau)\rangle^{(\alpha)}=\int \mathrm{d}\boldsymbol{k}\,R_{ij}^{(\alpha)}(\boldsymbol{k},\tau) \tag{1.6}$$

where

$$R_{ij}^{(\alpha)}(\boldsymbol{k},\tau)=\chi_\alpha\langle u_i^{(\alpha)}(\boldsymbol{k},t)u_j^{(\alpha)}(-\boldsymbol{k},t+\tau)\rangle^{(\alpha)}. \tag{1.7}$$

Clearly

$$F_{ij}^{(\alpha)}(\boldsymbol{k},t)=\tfrac{1}{2}R_{ij}^{(\alpha)}(\boldsymbol{k},0). \tag{1.8}$$

In the course of the analysis, a quantity

$$\eta_{ij}^{(\alpha)}\equiv\int_0^\infty \mathrm{d}\tau\,R_{ij}^{(\alpha)}(\tau) \tag{1.9}$$

will emerge which has both the dimensions and the function of a momentum transport coefficient; as it originates with the turbulent motion it will be called the eddy viscosity of rank α. Using (1.6) in (1.9) gives

$$\eta_{ij}^{(\alpha)}=\int \mathrm{d}\boldsymbol{k}\,\eta_{ij}^{(\alpha)}(\boldsymbol{k}) \tag{1.10}$$

where

$$\eta_{ij}^{(\alpha)}(\boldsymbol{k})\equiv\int_0^\infty \mathrm{d}\tau\,R_{ij}^{(\alpha)}(\boldsymbol{k},\tau). \tag{1.11}$$

From the definition (1.10) it is seen that $\eta_{ij}^{(\alpha)}$ is obtained by integrating a correlation function over the full range of its argument. Consequently, the eddy viscosity is characterized (like molecular transport coefficients) by a scale which is much larger than the scales of the microscopic processes from which it arises. The principal contribution to $\eta_{ij}^{(\alpha)}$ in (1.10) will come from large scales or small wavenumbers. This means that $\eta_{ij}^{(\alpha)}(\boldsymbol{k})$ is strongly peaked around $k=0$ and the approximation will be made that

$$\int \mathrm{d}\boldsymbol{k}\,f(\boldsymbol{k})\eta_{ij}^{(\alpha)}(\boldsymbol{k})\cong f(0)\int \mathrm{d}\boldsymbol{k}\,\eta_{ij}^{(\alpha)}(\boldsymbol{k})=f(0)\eta_{ij}^{(\alpha)} \tag{1.12}$$

where $f(\boldsymbol{k})$ is any continuous, integrable function of $\boldsymbol{k}$.

It might be worthwhile to note some of the results which have been obtained by Tchen (1973) in the case of homogeneous, isotropic turbulence. In the universal equilibrium range, his spectra are

inertial subrange ($k<k_\nu$): $\quad F(k)=1{\cdot}59\epsilon^{2/3}k^{-5/3}$ (1.13)

dissipation subrange ($k_\nu<k<k_c$): $\quad F(k)=0{\cdot}15(\epsilon/\nu^2)^2k^{-7}$ (1.14)

viscous cutoff subrange ($k\sim k_c$): $\quad F(k)=0{\cdot}15(\epsilon/\nu^2)^2k^{-7}\exp(-k/k_c)$ (1.15)

where $k_\nu=0{\cdot}17k_s$, $k_c=0{\cdot}74k_s$ and $k_s=(\epsilon/\nu^3)^{1/4}$. The Kolmogorov and Heisenberg spectra are obtained in the inertial and dissipation subranges, respectively. In the higher wavenumber portion of the dissipation subrange (here called the 'viscous cutoff'), a Heisenberg spectrum with an exponential tail is found. Results based on these spectra are plotted in figure 1, which follows Kraichnan's (1966) method of presentation. The

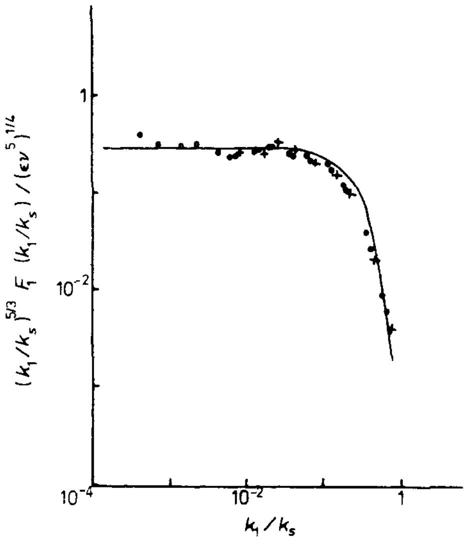

Figure 1. Spectra of Tchen (1973) (full curve) compared with the data of Grant *et al* (1962) in inertial and dissipation subranges.

experimental points are based on the work of Grant *et al* (1962). F_1 is the one-dimensional spectrum corresponding to the three-dimensional spectrum F.

The transition spectrum between the dissipation and inertial subranges has been calculated by numerically integrating Tchen's equations. A good agreement with experimental data is observed, comparable to that obtained by Kraichnan using the abridged Lagrangian direct interaction approximation. The numerical value of 1·59 for the Kolmogorov constant (3-dimensional spectrum) is lower than Kraichnan's (1·77), who observes that his is rather high, due, at least in part, to computational reasons. The 1·59 value compares favourably with the Kolmogorov constant based on the observations of Gibson (1963), Gibson and Schwatz (1963), Wyngaard and Pao (1971) and Wyngaard and Coté (1971). It is somewhat higher, however, than the value of 1·47 found by Grant *et al* (1962).

2. Turbulent hierarchy of equations

In its non-turbulent state the atmosphere is assumed to be at rest with a temperature $T_0(\boldsymbol{x}, t)$ which is maintained by external means that do not form part of this problem. T_0 is assumed to have a constant gradient. In its turbulent state, there will be an additional fluctuating component θ of the temperature so that the total temperature T is

$$T = T_0 + \theta.$$

There will also be a turbulent velocity $\boldsymbol{u}$. The equations governing $\boldsymbol{u}$ and θ are the conservation of momentum equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j\, \partial x_j} - \alpha\theta\hat{g}_i, \tag{2.1}$$

the energy equation

$$\frac{\partial \theta}{\partial t}+u_j\frac{\partial \theta}{\partial x_j}=\lambda\frac{\partial^2\theta}{\partial x_j \partial x_j}+u_j\beta_j \tag{2.2}$$

and the continuity equation

$$\frac{\partial u_j}{\partial x_j}=0, \tag{2.3}$$

where ρ_0 is the static density, p is the total pressure minus static pressure, ν is the kinematic viscosity, λ is the thermal diffusivity, $\hat{\boldsymbol{g}}$ is the unit vector in the direction of gravitational acceleration, β is the lapse rate, equal to $-\partial T_0/\partial x_j$ and $\alpha=-N^2/\beta$, where N is the Brunt–Vaisala frequency.

In deriving equations (2.1)–(2.3), the Boussinesq approximation was used and the Rayleigh dissipation term was neglected in (2.2), since it is assumed that the molecular viscosity will act primarily to dissipate the kinetic energy with the dissipation range of the thermal energy spectrum being governed by the thermal diffusivity. Similar equations have been utilized by Monin (1962) and by Lumley and Panofsky (1964).

The cascade expansions of velocity, pressure and temperature, all of form (1.1), are introduced into (2.1)–(2.3) followed by application of the averaging operator $\langle\ \rangle^{(\alpha)}$. The result for $\alpha=0,1,2,\ldots$ is an hierarchy of turbulence equations, each of (2.1)–(2.3) generating its own sequence of coupled equations. The first two equations in each of the hierarchies coming from (2.1) and (2.3) are as follows:

$$\frac{\mathrm{d}u_i^{(0)}}{\mathrm{d}t}=-\frac{1}{\rho_0}\frac{\partial p^{(0)}}{\partial x_i}-\left\langle u_j^{(1)}\frac{\partial u_i^{(1)}}{\partial x_j}\right\rangle^{(1)}+\nu\frac{\partial^2 u_i^{(0)}}{\partial x_j\partial x_j}-\alpha\hat{g}_i\theta \tag{2.4}$$

$$\frac{\mathrm{d}u_i^{(1)}}{\mathrm{d}t}=-\frac{1}{\rho_0}\frac{\partial p^{(1)}}{\partial x_i}-\left\langle u_j^{(2)}\frac{\partial u_i^{(2)}}{\partial x_j}\right\rangle^{(2)}-u_j^{(1)}\frac{\partial u_i^{(0)}}{\partial x_j}+\nu\frac{\partial^2 u_i^{(1)}}{\partial x_j\partial x_j}-\alpha\hat{g}_i\theta \tag{2.5}$$

$$\frac{\mathrm{d}\theta^{(0)}}{\mathrm{d}t}=u_j^{(0)}\beta_j-\left\langle u_j^{(1)}\frac{\partial \theta^{(1)}}{\partial x_j}\right\rangle^{(1)}+\lambda\frac{\partial^2\theta^{(0)}}{\partial x_j\partial x_j} \tag{2.6}$$

$$\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}t}=u_j^{\prime(1)}\left(\beta_j-\frac{\partial \theta^{(0)}}{\partial x_j}\right)-\left\langle u_j^{(2)}\frac{\partial \theta^{(2)}}{\partial x_j}\right\rangle^{(2)}+\lambda\frac{\partial^2\theta^{(1)}}{\partial x_j\partial x_j} \tag{2.7}$$

where the Lagrangian operator is

$$\frac{\mathrm{d}}{\mathrm{d}t}[\text{rank }\alpha]\equiv\left(\frac{\partial}{\partial t}+[u_j^{(0)}+u_j^{(1)}+\ \ldots\ +u_j^{(\alpha)}]\frac{\partial}{\partial x_j}\right)[\text{rank }\alpha].$$

Equations (2.3) gives for each rank

$$\frac{\partial u_j^{(\alpha)}}{\partial x_j}=0. \tag{2.8}$$

Fourier transforming (2.4)–(2.8) and using (2.8) to eliminate the pressure in the usual manner leads to the following equations for the Fourier transforms $\boldsymbol{u}^{(\alpha)}(\boldsymbol{k},t)$ and

$\theta^{(\alpha)}(\boldsymbol{k}, t)$ of $\boldsymbol{u}^{(\alpha)}(\boldsymbol{x}, t)$ and $\theta^{(\alpha)}(\boldsymbol{x}, t)$, respectively:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_i^{(0)}(\boldsymbol{k}, t) = -\int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k_j' \Delta_{il}(\boldsymbol{k}) \langle u_l^{(1)}(\boldsymbol{k}', t) u_j^{(1)}(\boldsymbol{k}-\boldsymbol{k}', t)\rangle^{(1)} - \nu k^2 u_i^{(0)}(\boldsymbol{k}, t) - \alpha \Delta_{il}(\boldsymbol{k}) \hat{g}_l \theta^{(0)}(\boldsymbol{k}, t) \tag{2.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u_i^{(1)}(\boldsymbol{k}, t) = -\int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k_j' \Delta_{il}(\boldsymbol{k}) [\langle u_l^{(2)}(\boldsymbol{k}', t) u_j^{(2)}(\boldsymbol{k}-\boldsymbol{k}', t)\rangle^{(2)} + u_l^{(0)}(\boldsymbol{k}', t) u_j^{(1)}(\boldsymbol{k}-\boldsymbol{k}', t)] - \nu k^2 u_i^{(1)}(\boldsymbol{k}, t) - \alpha \Delta_{il}(\boldsymbol{k}) \hat{g}_l \theta^{(1)}(\boldsymbol{k}, t) \tag{2.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \theta^{(0)}(\boldsymbol{k}, t) = -\int \mathrm{d}\boldsymbol{k}'\mathrm{i}k_j' \langle \theta^{(1)}(\boldsymbol{k}', t) u_j^{(1)}(\boldsymbol{k}-\boldsymbol{k}', t)\rangle^{(1)} - \lambda k^2 \theta^{(0)}(\boldsymbol{k}, t) + u_j^{(0)}(\boldsymbol{k}, t)\beta_j \tag{2.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \theta^{(1)}(\boldsymbol{k}, t) = -\int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k_j' [\langle \theta^{(2)}(\boldsymbol{k}', t) u_j^{(2)}(\boldsymbol{k}-\boldsymbol{k}', t)\rangle^{(2)} + \theta^{(0)}(\boldsymbol{k}', t) u_j^{(1)}(\boldsymbol{k}-\boldsymbol{k}', t)] - \lambda k^2 \theta^{(1)}(\boldsymbol{k}, t) + u_j^{(1)}(\boldsymbol{k}, t)\beta_j \tag{2.12}$$

where

$$\Delta_{ij}(\boldsymbol{k}) \equiv \delta_{ij} - k_i k_j / k^2.$$

A hierarchy of coupled equations is also found in the usual formulation of turbulence problems. However, this hierarchy results from an attempt to find a closed system by generating equations for successively higher order correlations of the fluctuating variables. The hierarchies found here result from the fact that any particular scale of the turbulence is linked dynamically to neighbouring scales. A more intricate coupling will be found in the next section when the equations governing energy (second order correlation) are derived.

3. Equations for energy spectra

The equation governing the velocity energy spectrum tensor $F_{ij}^{(0)}$ is easily obtained using the definition (1.4) and equation (2.9). Integrating the equation so derived over the cutoff wavenumbers for rank zero, with $k_{-1}=0$ and denoting k_0 by κ, see (1.3), where κ is arbitrary gives

$$\frac{\partial}{\partial t} \int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, F_{ij}(\boldsymbol{k}, t) = -T_{ij}(\kappa, t) + B_{ij}(\kappa, t) - 2\nu \int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 F_{ij}(\boldsymbol{k}, t) \tag{3.1}$$

where

$$T_{ij} \equiv \tfrac{1}{2}\chi_0 \int \mathrm{d}\boldsymbol{k}' \int \mathrm{d}\boldsymbol{k}''\, \mathrm{i}k_n'' \Big(\Delta_{il}(\boldsymbol{k}') \Big\langle u_j^{(0)}(-\boldsymbol{k}', t) \langle u_l^{(1)}(\boldsymbol{k}'', t) u_n^{(1)}(\boldsymbol{k}'-\boldsymbol{k}'', t)\rangle^{(1)} \Big\rangle^{(0)} + \Delta_{jl}(\boldsymbol{k}') \Big\langle u_i^{(0)}(\boldsymbol{k}', t) \langle u_l^{(1)}(\boldsymbol{k}'', t) u_n^{(1)}(-\boldsymbol{k}'-\boldsymbol{k}'', t)\rangle^{(1)} \Big\rangle^{(0)} \Big) \tag{3.2}$$

$$B_{ij} \equiv -\tfrac{1}{2}\alpha \hat{g}_l \chi_0 \int \mathrm{d}\boldsymbol{k}' [\Delta_{il}(\boldsymbol{k}') \langle u_j^{(0)}(-\boldsymbol{k}', t) \theta^{(0)}(\boldsymbol{k}', t)\rangle^{(0)} + \Delta_{jl}(\boldsymbol{k}') \langle u_i^{(0)}(\boldsymbol{k}', t) \theta^{(0)}(-\boldsymbol{k}', t)\rangle^{(0)}]. \tag{3.3}$$

Following a similar procedure, starting with equation (2.11), an equation for the energy spectrum for thermal fluctuations, defined analogously to (1.4) as

$$H^{(\alpha)}(\boldsymbol{k}, t) \equiv \tfrac{1}{2}\chi_\alpha \langle \theta^{(\alpha)}(\boldsymbol{k}, t)\theta^{(\alpha)}(-\boldsymbol{k}, t)\rangle^{(\alpha)},$$

can be derived as follows:

$$\frac{\partial}{\partial t}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, H(\boldsymbol{k}, t) = -W(\kappa, t) + C(\kappa, t) - 2\lambda \int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 H(\boldsymbol{k}, t) \tag{3.4}$$

where

$$W \equiv \tfrac{1}{2}\chi_0 \int \mathrm{d}\boldsymbol{k}' \int \mathrm{d}\boldsymbol{k}''\, \mathrm{i}k_j''\Big(\Big\langle \theta^{(0)}(\boldsymbol{k}', t)\langle \theta^{(1)}(\boldsymbol{k}'', t) u_j^{(1)}(-\boldsymbol{k}'-\boldsymbol{k}'', t)\rangle^{(1)}\Big\rangle^{(0)}$$

$$+\Big\langle \theta^{(0)}(-\boldsymbol{k}', t)\langle \theta^{(1)}(\boldsymbol{k}'', t) u_j^{(1)}(\boldsymbol{k}'-\boldsymbol{k}'', t)\rangle^{(1)}\Big\rangle^{(0)}\Big) \tag{3.5}$$

$$C \equiv \tfrac{1}{2}\beta_j\chi_0 \int \mathrm{d}\boldsymbol{k}''\, [\langle \theta^{(0)}(\boldsymbol{k}'', t) u_j^{(0)}(-\boldsymbol{k}'', t)\rangle^{(0)} + \langle \theta^{(0)}(-\boldsymbol{k}'', t) u_j^{(0)}(\boldsymbol{k}'', t)\rangle^{(0)}]. \tag{3.6}$$

$F_{ij}^{(\alpha)}$ and $H^{(\alpha)}$ were replaced by F and H, respectively, in (3.1) and (3.4) by using the property (1.5). In (3.1) and (3.4) the arbitrary upper cutoff wavenumber, κ, is the independent variable. Each of these equations has the form

time rate of change of energy spectrum in wavenumber range 0 to κ

= rate of nonlinear (or inertial) transfer across spectrum

+ rate of generation or transfer between the F and H spectra due to coupling

+ rate of molecular dissipation.

Although both the B and C terms act as energy sources for their respective spectra, B is in reality a coupling term. Before (3.1) and (3.4) can yield solutions for the spectra, the transfer terms T and W, and the energy generation/coupling terms C and B, must be expressed in terms of F, H and/or other quantities which can be calculated, i.e. the system must be closed.

The calculations leading to an evaluation of the generation/coupling terms are lengthy and are described in appendix 1. The results are

$$B_{ij} = -\tfrac{1}{2}\alpha\beta_m(\hat{g}_i\eta_{mj}^{(0)} + \hat{g}_j\eta_{mi}^{(0)}) \tag{3.7}$$

$$C = \beta_j\beta_m\eta_{mj}^{(0)} \tag{3.8}$$

where the notation for the eddy viscosity introduced in (1.9) has been used.

The calculation of the transfer term T_{ij} follows the corresponding method of Tchen (1973) up to the point where (3.9) results (see his paper for details). The difference here is that an extra term, arising from the lapse rate, must be taken into account, but it is not difficult to show that this term does not contribute to the inertial transfer. Hence

$$T_{ij} = 2\eta_{ns}^{(1)}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k_s k_n F_{ir}(\boldsymbol{k}, t)\Delta_{jr}(\boldsymbol{k}). \tag{3.9}$$

The method of calculating the other transfer term W is basically the same as that for T_{ij} except that the starting point is the corresponding equation for $\theta^{(1)}$. The result, which is analogous to (3.9), is

$$W = 2\eta_{ns}^{(1)} \int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k_s k_n H(\boldsymbol{k}, t). \tag{3.10}$$

Substitution of (3.7)–(3.10) into (3.1) and (3.4) yields the following equations for the energy spectra:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, F_{ij}(\boldsymbol{k}, t) &= -\tfrac{1}{2}\alpha\beta_m(\hat{g}_i\eta_{mj}^{(0)} + \hat{g}_j\eta_{mi}^{(0)}) - 2\nu\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 F_{ij}(\boldsymbol{k}, t)\\ &\quad - 2\eta_{ns}^{(1)}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k_s k_n \Delta_{jr}(\boldsymbol{k}) F_{ir}(\boldsymbol{k}, t)\end{aligned} \tag{3.11}$$

$$\frac{\partial}{\partial t}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, H(\boldsymbol{k}, t) = \beta_j\beta_m\eta_{mj}^{(0)} - 2(\lambda\delta_{ns} + \eta_{ns}^{(1)})\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k_s k_n H(\boldsymbol{k}, t). \tag{3.12}$$

In these equations the eddy viscosity has not yet been calculated so the system is not actually closed. This will be accomplished in the next section by formulating and solving equations for the eddy viscosity.

4. The eddy viscosity

Instead of calculating the eddy viscosity directly, the correlation $R_{ij}^{(\alpha)}(\boldsymbol{k}, \tau)$ will be found and used to determine $\eta_{ij}^{(\alpha)}$. Multiplying (2.10) by $\chi_1 u_j^{(1)}(-\boldsymbol{k}, t')$, averaging and assuming a locally homogeneous turbulence gives

$$\frac{\partial}{\partial\tau} R_{ij}^{(1)}(\boldsymbol{k}, \tau) = L_{ij}^{(1)}(\boldsymbol{k}, \tau) + Q_{ij}^{(1)}(\boldsymbol{k}, \tau) - \alpha\hat{g}_l\Delta_{il}(\boldsymbol{k})\mathscr{R}_j^{(1)}(\boldsymbol{k}, \tau) \tag{4.1}$$

where $\tau \equiv t' - t$ and

$$\mathscr{R}_j^{(1)}(\boldsymbol{k}, \tau) \equiv \chi_1\langle u_j^{(1)}(-\boldsymbol{k}, t)\theta^{(1)}(\boldsymbol{k}, t+\tau)\rangle^{(1)}$$

$$L_{ij}^{(1)}(\boldsymbol{k}, \tau) \equiv -\chi_1 \int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k'_m\Delta_{il}(\boldsymbol{k})\Big\langle u_j^{(1)}(-\boldsymbol{k}, t+\tau)\langle u_l^{(2)}(\boldsymbol{k}', t)u_m^{(2)}(\boldsymbol{k}-\boldsymbol{k}', t)\rangle^{(2)}\Big\rangle^{(1)}$$

$$Q_{ij}^{(1)}(\boldsymbol{k}, \tau) \equiv -\chi_1 \int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k'_m\Delta_{il}(\boldsymbol{k})u_l^{(0)}(\boldsymbol{k}', t)\langle u_j^{(1)}(-\boldsymbol{k}, t+\tau)u_m^{(1)}(\boldsymbol{k}-\boldsymbol{k}', t)\rangle^{(1)}.$$

The viscous term in (2.10) has been neglected since the eddy viscosity is a turbulent, and not molecular, transport property. The equation governing $\mathscr{R}_j^{(1)}(\boldsymbol{k}, \tau)$ is found by treating equation (2.12) in a similar manner, and is

$$\frac{\partial}{\partial\tau}\mathscr{R}_j^{(1)}(\boldsymbol{k}, \tau) = \mathscr{L}_j^{(1)}(\boldsymbol{k}, \tau) + \mathscr{Q}_j^{(1)}(\boldsymbol{k}, \tau) + \beta_m R_{mj}^{(1)}(\boldsymbol{k}, \tau) \tag{4.2}$$

where

$$\mathscr{L}_j^{(1)}(\boldsymbol{k},\tau) \equiv -\chi_1 \int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k'_m \Big\langle u_j^{(1)}(-\boldsymbol{k}, t+\tau)\langle\theta^{(2)}(\boldsymbol{k}',t)u_m^{(2)}(\boldsymbol{k}-\boldsymbol{k}',t)\rangle^{(2)}\Big\rangle^{(1)}$$

$$\mathscr{Q}_j^{(1)}(\boldsymbol{k},\tau) \equiv -\chi_1 \int \mathrm{d}\boldsymbol{k}'\, \mathrm{i}k'_m \theta^{(0)}(\boldsymbol{k}',t)\langle u_j^{(1)}(-\boldsymbol{k}, t+\tau)u_m^{(1)}(\boldsymbol{k}-\boldsymbol{k}',t)\rangle^{(1)}.$$

The evaluation of $L^{(1)}$, $Q^{(1)}$, $\mathscr{L}^{(1)}$ and $\mathscr{Q}^{(1)}$ is given in appendix 2. The results are

$$\begin{aligned} L_{ij}^{(1)}(\boldsymbol{k},\tau) &= -\Omega^{(2)}(\boldsymbol{k})R_{ij}^{(1)}(\boldsymbol{k},\tau) \\ Q_{ij}^{(1)}(\boldsymbol{k},\tau) &= \mathscr{Q}_j^{(1)}(\boldsymbol{k},\tau) = 0 \\ \mathscr{L}_j^{(1)}(\boldsymbol{k},\tau) &= -\Omega^{(2)}(\boldsymbol{k})\mathscr{R}_j^{(1)}(\boldsymbol{k},\tau) \end{aligned} \tag{4.3}$$

where

$$\Omega^{(\alpha)}(\boldsymbol{k}) \equiv k_i k_j \eta_{ij}^{(\alpha)} \tag{4.4}$$

is the turbulent relaxation frequency for rank α. Substituting (4.3) into (4.1) and (4.2) gives the following system of simultaneous equations to be solved for $R_{ij}^{(1)}(\boldsymbol{k},\tau)$:

$$\frac{\partial}{\partial\tau} R_{ij}^{(1)}(\boldsymbol{k},\tau) = -\Omega^{(2)}(\boldsymbol{k})R_{ij}^{(1)}(\boldsymbol{k},\tau) - \alpha\hat{g}_l\Delta_{il}(\boldsymbol{k})\mathscr{R}_j^{(1)}(\boldsymbol{k},\tau) \tag{4.5}$$

$$\frac{\partial}{\partial\tau}\mathscr{R}_j^{(1)}(\boldsymbol{k},\tau) = -\Omega^{(2)}(\boldsymbol{k})\mathscr{R}_j^{(1)}(\boldsymbol{k},\tau) + \beta_m R_{mj}^{(1)}(\boldsymbol{k},\tau). \tag{4.6}$$

Choosing the x_3 axis to point vertically upward and a simple vertical temperature gradient model $\beta_i = \beta\delta_{i3}$ gives the following solution for $R_{ij}^{(1)}$:

$$R_{ij}^{(1)}(\boldsymbol{k},\tau) = A_{ij}\,\mathrm{e}^{m_+\tau} + C_{ij}\,\mathrm{e}^{m_-\tau}$$

where $m_\pm = -\Omega^{(2)} \pm \sqrt{N^2\Delta_{33}}$. The solution appropriate for an unstable atmosphere (negative Brunt–Vaisala frequency) and satisfying the condition that

$$\lim_{\tau\to\infty} R_{ij}^{(1)}(\boldsymbol{k},\tau) = 0$$

is chosen. This gives

$$R_{ij}^{(1)}(\boldsymbol{k},\tau) = R_{ij}^{(1)}(\boldsymbol{k},0)\exp[-(\Omega^{(2)} + k_{\mathrm{H}}\sigma/k)\tau] \tag{4.7}$$

where

$$N^2 \equiv -\sigma^2 \qquad (\sigma > 0)$$
$$k_{\mathrm{H}}^2 \equiv k_1^2 + k_2^2.$$

Substituting (4.7) into (1.11) and using (1.8) gives

$$\eta_{ij}^{(1)}(\boldsymbol{k}) = 2\frac{F_{ij}^{(1)}(\boldsymbol{k})}{\Omega^{(2)} + k_{\mathrm{H}}\sigma/k}.$$

A further substitution into (1.10) and using (1.5) gives

$$\eta_{ij}^{(1)} = 2\int_{k=\kappa}^{k_1} \mathrm{d}\boldsymbol{k}\frac{F_{ij}(\boldsymbol{k})}{\Omega^2 + k_\mathrm{H}\sigma/k}. \tag{4.8}$$

In view of the definition (4.4) of $\Omega^{(2)}$, it is seen that (4.8) is actually one member of a hierarchy which expresses $\eta_{ij}^{(\alpha)}$ in terms of $\eta_{ij}^{(\alpha+1)}$. This hierarchy is truncated, and hence the system closed, by choosing $k_1 \gg \kappa$ so that effectively the limits of integration are from κ to infinity and by assuming $\Omega^{(2)} \cong \Omega^{(1)}$. Thus

$$\eta_{ij}^{(1)} = 2\int_{k=\kappa}^{\infty} \mathrm{d}\boldsymbol{k}\frac{F_{ij}(\boldsymbol{k})}{\Omega^{(1)} + k_\mathrm{H}\sigma/k}. \tag{4.9}$$

Equations (4.5) and (4.6) can also be used to give information about relationships among the elements $R_{ij}^{(1)}(\boldsymbol{k}, 0)$. Solving these equations for $R_{33}^{(1)}(\boldsymbol{k}, 0)$ first and then for all other elements $R_{ij}^{(1)}(\boldsymbol{k}, 0)$ gives the latter in terms of $R_{33}^{(1)}(\boldsymbol{k}, 0)$. Thus, $R_{ij}^{(1)}(\boldsymbol{k}, 0)$ can be related to $R_{33}^{(1)}(\boldsymbol{k}, 0)$ or, equivalently, to the trace $R_{ii}^{(1)}(\boldsymbol{k}, 0)$. Using (1.8), it is found that

$$F_{11}(\boldsymbol{k}) = \left(\frac{k_1 k_3}{k k_\mathrm{H}}\right)^2 F_{ii}(\boldsymbol{k}) \tag{4.10a}$$

$$F_{22}(\boldsymbol{k}) = \left(\frac{k_2 k_3}{k k_\mathrm{H}}\right)^2 F_{ii}(\boldsymbol{k}) \tag{4.10b}$$

$$F_{33}(\boldsymbol{k}) = \left(\frac{k_\mathrm{H}}{k}\right)^2 F_{ii}(\boldsymbol{k}) \tag{4.10c}$$

$$F_{13}(\boldsymbol{k}) = -\frac{k_1 k_3}{k^2} F_{ii}(\boldsymbol{k}) \tag{4.10d}$$

$$F_{23}(\boldsymbol{k}) = -\frac{k_2 k_3}{k^2} F_{ii}(\boldsymbol{k}). \tag{4.10e}$$

5. Solutions for energy spectra

In view of (4.10), the equation for F_{ii} is required. Contracting i and j in (3.11) leads to an equation involving T_{ii}, where

$$T_{ii} = 2\eta_{ns}^{(1)} \int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\; k_s k_n [F_{ii}(\boldsymbol{k}, t) - (k_i k_r / k^2) F_{ir}(\boldsymbol{k}, t)]$$

in which the definition of $\Delta_{ir}(\boldsymbol{k})$ has been used. The scalar trace $F_{ii}(\boldsymbol{k}, t)$ is associated with the turbulent kinetic energy and, as such, should depend primarily upon the magnitude k of the wavevector $\boldsymbol{k}$ only. The elements $F_{ir}(\boldsymbol{k}, t)$ are more sensitive to the anisotropy induced by the temperature gradient so the term involving F_{ir} is more difficult to treat. For the purpose of estimating the contribution of this to T_{ii}, it is also assumed that $F_{ir}(\boldsymbol{k}, t) = F_{ir}(k, t)$. Furthermore, it is expected that at the small scales which contribute to the eddy viscosity $\eta_{ns}^{(1)}$, the turbulence is isotropic so that

$$\eta_{ns}^{(1)} = \eta^{(1)}\delta_{ns} \tag{5.1}$$

where the isotropic eddy viscosity is

$$\eta^{(1)} = \tfrac{1}{3}\eta_{ii}^{(1)}.$$

These assumptions give the result

$$T_{ii} = 2\times\tfrac{2}{3}\eta^{(1)}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 4\pi k^2 F_{ii}(k,t). \tag{5.2}$$

Using the simple temperature gradient model and (5.2), the contracted form of (3.11) is

$$\frac{\partial}{\partial t}\int_{k=0}^{\kappa} \mathrm{d}k\; 4\pi k^2 F_{ii}(k,t) = \sigma^2\eta_{33}^{(0)} - 2(\nu + \tfrac{2}{3}\eta^{(1)})\int_{k=0}^{\kappa} \mathrm{d}k\, k^2 4\pi k^2 F_{ii}(k,t). \tag{5.3}$$

Similarly, assuming that $H(\boldsymbol{k},t) = H(k,t)$ gives for (3.12)

$$\frac{\partial}{\partial t}\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\; 4\pi k^2 H(k,t) = \beta^2\eta_{33}^{(0)} - 2(\lambda + \tfrac{2}{3}\eta^{(1)})\int_{k=0}^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 4\pi k^2 H(k,t). \tag{5.4}$$

The eddy viscosity $\eta^{(1)}$ is obtained from (4.9) as

$$\eta^{(1)} \equiv \tfrac{1}{3}\eta_{ii}^{(1)} = \frac{2}{3}\int_{k=\kappa}^{\infty} \mathrm{d}\boldsymbol{k}\,\frac{F_{ii}(\boldsymbol{k})}{\Omega^{(1)} + k_{\mathrm{H}}\sigma/k}.$$

Since at the higher wavenumbers and smaller scales which contribute to $\eta^{(1)}$ the turbulence is nearly isotropic, a good approximation is that

$$\Omega^{(1)} + k_{\mathrm{H}}\sigma/k \cong k^2\eta^{(1)} + \sigma$$

so that $\eta^{(1)}$ becomes

$$\eta^{(1)} = \frac{2}{3}\int_{k=\kappa}^{\infty} \mathrm{d}k\,\frac{4\pi k^2 F_{ii}(k)}{k^2\eta^{(1)} + \sigma}. \tag{5.5}$$

Using (1.10), since $k_{-1} = 0$, $k_0 = \kappa$ and $k_1 = \infty$, (5.5) and (4.10*c*), $\eta_{33}^{(0)}$ can be expressed in terms of $\eta^{(1)}$ as

$$\eta_{33}^{(0)} = \eta_{33}^{(0)}(\infty) - 2\eta^{(1)} \tag{5.6}$$

where $\eta_{33}^{(0)}(\infty) \equiv \lim_{\kappa\to\infty}\eta_{33}^{(0)}$.

Solutions of (5.3) and (5.4) will be sought for high wavenumbers. Taking the limit of both sides of each of these equations as κ becomes infinite, and then approximating the left-hand sides with these limiting values, gives the following spectral equations, valid for high wavenumbers,

$$2\sigma^2\eta^{(1)} + 2(\nu + \tfrac{2}{3}\eta^{(1)})\int_0^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 F(k) = \epsilon \tag{5.7}$$

$$2\beta^2\eta^{(1)} + 2(\lambda + \tfrac{2}{3}\eta^{(1)})\int_0^{\kappa} \mathrm{d}\boldsymbol{k}\, k^2 E(k) = \epsilon_\lambda \tag{5.8}$$

where

$$F(k) \equiv 4\pi k^2 F_{ii}(k) \qquad E(k) \equiv 4\pi k^2 H(k)$$

$$\epsilon = 2\nu \int_0^\infty dk\, k^2 F(k) \qquad \epsilon_\lambda = 2\lambda \int_0^\infty dk\, k^2 E(k).$$

F and E are the three-dimensional scalar energy spectra and ϵ and ϵ_λ are the dissipation rates of kinetic and thermal energies, respectively.

For each of the velocity and temperature spectra, there are wavenumber ranges defined by certain characteristic wavenumbers which are determined by the parameters which have physical significance in the problem. The characteristic wavenumbers for the velocity spectrum are as follows:

buoyancy wavenumber	$k_\sigma = (\sigma/\nu)^{1/2}$
dissipation wavenumber	$k_\nu = (\epsilon/\nu^3)^{1/4}$

and for the temperature spectrum:

thermal wavenumber	$k_\beta = (c_p\beta/\lambda^2)^{1/3}$
dissipation wavenumber	$k_\lambda = (c_p^2\epsilon_\lambda/\lambda^5)^{1/6}$

where c_p is the specific heat at constant pressure. The buoyancy and thermal wavenumbers characterize the scale at which energy production and coupling are important. The dissipation wavenumbers characterize the scales at which molecular transport coefficients dissipate the energy of the turbulent fluctuations. The energy production ranges are $k < k_\sigma$ and $k < k_\beta$, the inertial transfer subranges are $k_\sigma \ll k < k_\nu$ and $k_\beta \ll k < k_\lambda$ and the dissipation subranges are $k \gtrsim k_\nu$ and $k \gtrsim k_\lambda$. These last two subranges comprise the universal equilibrium range for each spectrum. For wavenumbers $k \gtrsim k_\sigma$ and $k \gtrsim k_\beta$ both the energy production and the inertial transfer are important; wavenumbers where this is true comprise the production-transfer subrange.

In general, the wavenumber scales associated with each of the spectra are shifted relative to each other, with the characteristic wavenumbers for the velocity spectrum being larger than those for the temperature spectrum in view of the large rate of dissipation of kinetic energy. Hence, solutions for the two spectra will be found for the cases summarized in table 1.

5.1. Cases A

In these cases the wavenumber is in the production-transfer range of the velocity spectrum. Hence the two terms describing these processes are the important terms in (5.7). Dropping the viscous term gives

$$2\sigma^2\eta^{(1)} + \tfrac{4}{3}\eta^{(1)} \int_0^\kappa dk\, k^2 F(k) = \epsilon. \tag{5.9}$$

Case A1. The equation resulting from (5.8) which is analogous to (5.9) is

$$2\beta^2\eta^{(1)} + \tfrac{4}{3}\eta^{(1)} \int_0^\kappa dk\, k^2 E(k) = \epsilon_\lambda. \tag{5.10}$$

Table 1. Cases in which solutions for spectra are found.

Wavenumber ranges for temperature spectrum	Wavenumber ranges for velocity spectrum: $k \gtrsim k_\sigma$ Production-transfer range	$k_\nu > k \gg k_\sigma$ Inertial subrange	$k > k_\nu$ Dissipation subrange
$k \gtrsim k_\beta$ Production-transfer range	Case A_1		
$k_\lambda > k \gg k_\beta$ Inertial subrange	Case A_2	Case B_1	
$k > k_\lambda$ Dissipation subrange	Case A_3	Case B_2	Case C

Differentiating (5.9) and (5.10) with respect to κ and neglecting $\mathrm{d}\eta^{(1)}/\mathrm{d}\kappa$ except when it occurs in conjunction with a factor like σ^2 or β^2 which may not be small gives

$$\tfrac{2}{3}\kappa^2\eta^{(1)}F+\sigma^2\frac{\mathrm{d}\eta^{(1)}}{\mathrm{d}\kappa}=0 \tag{5.11a}$$

$$\tfrac{2}{3}\kappa^2\eta^{(1)}E+\beta^2\frac{\mathrm{d}\eta^{(1)}}{\mathrm{d}\kappa}=0. \tag{5.11b}$$

The derivative of the eddy viscosity is small since $\eta^{(1)}$ is slowly varying in the production-transfer range. Differentiating (5.5) gives

$$\frac{\mathrm{d}\eta_1^{(1)}}{\mathrm{d}\kappa}=-\frac{2}{3}\frac{F}{\kappa^2\eta^{(1)}+\sigma}. \tag{5.12}$$

Combining (5.11) and (5.12) leads to

$$\psi^2+\sigma\psi-\sigma^2=0 \qquad E=\beta^2\frac{F}{\psi(\psi+\sigma)}$$

where $\psi\equiv\kappa^2\eta^{(1)}$. The positive solution for ψ is $\psi=\frac{1}{2}\sigma(\sqrt{5}-1)\cong0{\cdot}62\sigma$. Hence

$$\eta^{(1)}=0{\cdot}62\sigma\kappa^{-2} \tag{5.13}$$

$$F(\kappa)=3\sigma^2\kappa^{-3} \tag{5.14a}$$

$$E(\kappa)=3\beta^2\kappa^{-3}. \tag{5.14b}$$

The components of the velocity spectrum tensor are found from (4.10) and are

$$F_{11}(k)=\frac{3\sigma^2}{4\pi}\left(\frac{k_1k_3}{kk_{\mathrm{H}}}\right)^2k^{-5} \tag{5.15a}$$

$$F_{22}(k)=\frac{3\sigma^2}{4\pi}\left(\frac{k_2k_3}{kk_{\mathrm{H}}}\right)^2k^{-5} \tag{5.15b}$$

$$F_{33}(k)=\frac{3\sigma^2}{4\pi}\left(\frac{k_{\mathrm{H}}}{k}\right)^2k^{-5} \tag{5.15c}$$

$$F_{13}(k)=-\frac{3\sigma^2}{4\pi}\left(\frac{k_1k_3}{k^2}\right)k^{-5} \tag{5.15d}$$

$$F_{23}(k)=-\frac{3\sigma^2}{4\pi}\left(\frac{k_2k_3}{k^2}\right)k^{-5}. \tag{5.15e}$$

Case A2. Equation (5.8) in the universal equilibrium range, becomes

$$2(\lambda+\tfrac{2}{3}\eta^{(1)})\int_0^\kappa \mathrm{d}k\,k^2E(k)=\epsilon_\lambda \tag{5.16}$$

reflecting the fact that the production term does not contribute in this range. Dividing (5.16) by $\lambda+\eta^{(1)}$, differentiating and proceeding in the manner used to derive (5.11) gives

$$2\kappa^2E=-\frac{2}{3}\frac{\epsilon_\lambda}{(\lambda+\frac{2}{3}\eta^{(1)})^2}\frac{\mathrm{d}\eta^{(1)}}{\mathrm{d}\kappa}. \tag{5.17}$$

Equations (5.11a) and (5.12) remain the same; hence the solutions for $\eta^{(1)}$, F and F_{ij} are the same as in case A1. Substituting (5.12), (5.13) and (5.14a) into (5.17) gives the solution for E as

$$E(k)=0{\cdot}41\epsilon_\lambda\sigma\frac{k^{-3}}{(0{\cdot}41\sigma k^{-1}+\lambda k)^2}. \tag{5.18}$$

In the inertial subrange the thermal diffusivity can be neglected giving

$$E(k)=2{\cdot}44\frac{\epsilon_\lambda}{\sigma}k^{-1}. \tag{5.19}$$

Case A3. The solutions for $\eta^{(1)}$, F and F_{ij} are the same as in case A1. Equation (5.18) is also valid, but now, in the dissipation subrange, the molecular thermal diffusivity dominates, resulting in

$$E(k)=0{\cdot}41\frac{\sigma\epsilon_\lambda}{\lambda^2}k^{-5}. \tag{5.20}$$

5.2. Cases B

In the universal equilibrium range the production or coupling terms no longer contribute and (5.7) and (5.8) become

$$2(\nu+\tfrac{2}{3}\eta^{(1)})\int_0^\kappa \mathrm{d}k\,k^2F(k)=\epsilon \tag{5.21a}$$

$$2(\lambda+\tfrac{2}{3}\eta^{(1)})\int_0^\kappa \mathrm{d}k\,k^2E(k)=\epsilon_\lambda. \tag{5.21b}$$

Similarly the contribution from the Brunt–Vaisala frequency to the eddy viscosity in (5.5) is small, since this term arises from the energy production by the temperature gradient. Hence $\eta^{(1)}$ becomes

$$\eta^{(1)}=\tfrac{2}{3}\int_{\kappa}^{\infty}\mathrm{d}k\frac{F(k)}{k^2\eta^{(1)}}. \tag{5.22}$$

Dividing (5.21*a*, *b*) by $\nu+\eta^{(1)}$ or $\lambda+\eta^{(1)}$, as appropriate, then differentiating these equations as well as (5.22) with respect to κ and finally combining the results gives

$$\left(\frac{\eta^{(1)}}{\nu}\right)^3+3\left(\frac{\eta^{(1)}}{\nu}\right)^2+\frac{9}{4}\frac{\eta^{(1)}}{\nu}-\frac{1}{2}\frac{\epsilon}{\nu^3}\kappa^{-4}=0 \tag{5.23}$$

$$E(\kappa)=\frac{2}{9}\frac{\epsilon_\lambda}{(\lambda+\frac{2}{3}\eta^{(1)})^2}\frac{F(\kappa)}{\kappa^4\eta^{(1)}}. \tag{5.24}$$

In the inertial subrange $\eta^{(1)}/\nu\gg 1$, (5.23) becomes

$$\left(\frac{\eta^{(1)}}{\nu}\right)^3-\frac{1}{2}\frac{\epsilon}{\nu^3}\kappa^{-4}=0$$

giving

$$\eta^{(1)}=\frac{1}{2^{1/3}}\epsilon^{1/3}\kappa^{-4/3} \tag{5.25}$$

$$F(\kappa)=1{\cdot}26\epsilon^{2/3}\kappa^{-5/3}. \tag{5.26}$$

Case B1. When the temperature spectrum is in the inertial subrange, the molecular thermal diffusivity can be neglected in (5.24), i.e.,

$$\lambda+\eta^{(1)}\cong\eta^{(1)}.$$

Using (5.25) and (5.26) gives

$$E(\kappa)=1{\cdot}26\epsilon_\lambda\epsilon^{-1/3}\kappa^{-5/3}. \tag{5.27}$$

Case B2. In the dissipation subrange the thermal diffusivity dominates the eddy viscosity in (5.24). Again using (5.25) and (5.26), the solution for the temperature spectrum is found to be

$$E(\kappa)=0{\cdot}35\frac{\epsilon_\lambda\epsilon^{1/3}}{\lambda^2}\kappa^{-13/3}. \tag{5.28}$$

5.3. *Case C*

In the dissipation subrange $\eta^{(1)}/\nu\ll 1$, (5.23) becomes approximately

$$\frac{\eta^{(1)}}{\nu}-0{\cdot}22\frac{\epsilon}{\nu^3}\kappa^{-4}=0.$$

Hence

$$\eta^{(1)}=0{\cdot}22\frac{\epsilon}{\nu^2}\kappa^{-4} \tag{5.29}$$

$$F(\kappa)=0.30\frac{\epsilon^2}{\nu^4}\kappa^{-7}. \tag{5.30}$$

In (5.24), the molecular term will dominate the eddy viscosity, so using (5.30) gives

$$E(\kappa) = 0{\cdot}30 \frac{\epsilon\epsilon_\lambda}{(\nu\lambda)^2} \kappa^{-7}. \tag{5.31}$$

6. Conclusion and comparison with other theories

The types of spectra which have been found in this paper have been observed by Kao and Wendell (1970) and by Kao (1970). Although the interpretation of the observed spectra is complicated by the fact that wind shear is also present, many of the features predicted in § 5 can be seen. Two examples are given in figures 2 and 3. Figure 1 shows the spectra of the zonal velocity and of the temperature measured at the 200 mb level and at latitude 40°N in the winter of 1964. The velocity spectrum is in the production-transfer subrange while the temperature spectrum passes through all three of its subranges (production-transfer, inertial and dissipation), as described in cases A.

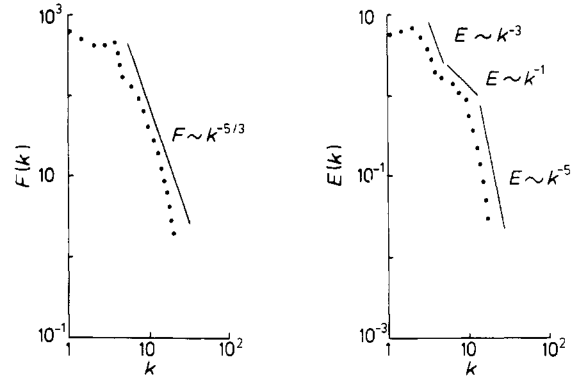

Figure 2. Spectra of zonal velocity and temperature as measured by Kao and Wendell (1970) and by Kao (1970) (open circles) in winter, 1964, compared with spectra derived in § 5 (straight lines).

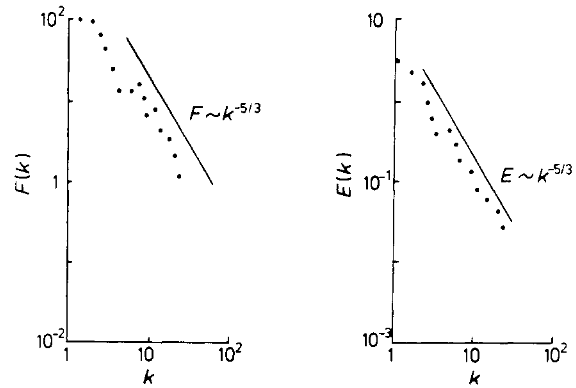

Figure 3. Spectra of zonal velocity and temperature as measured by Kao and Wendell (1970) and by Kao (1970) (open circles) in summer, 1964, compared with spectra derived in § 5 (straight lines).

Figure 2 shows the same spectra measured at the 200 mb level at latitude 20°N in the summer of 1964. In this instance both spectra display a $-\frac{5}{3}$ power law, indicating that both are in the inertial subrange which is described in case B1.

As can be seen from § 5, the theory presented here gives the Kolmogorov and Heisenberg velocity spectra in the inertial and viscous subranges, respectively. This is because the effects of the temperature gradient and buoyancy have been assumed to be confined to the production-transfer subrange exclusively. The Kolmogorov constant, which is found to be 1·26, is somewhat lower than most reported values. For example, the measurement of Gibson (1963) and Gibson and Schwartz (1963) lie in the range from 1·3 to 1·6. Grant *et al* (1962) find a value of 1·47. A lower value has been obtained here than in Tchen's (1973) work because a slightly different method of approximating the result of a complicated integration in wavenumber space which is involved in evaluating T_{ii} has been used (see equation (5.2)). The simplification used in this paper should lead to an estimate of the maximum value of the numerical factor involved. Consequently, a more precise evaluation would be expected to increase the Kolmogorov constant.

Equation (5.5) can be solved to give an explicit expression for the eddy viscosity in the inertial subrange, where $\eta^{(1)}+\sigma \cong \eta^{(1)}$, which is

$$\eta^{(1)}=\left(\frac{4}{3}\int_{k=\kappa}^{\infty} \mathrm{d}k\,\frac{F(k)}{k^2}\right)^{1/2}. \tag{6.1}$$

This expression is different from the classical eddy viscosity of Heisenberg (1948). However, (6.1) is a special case of the generalized form for the eddy viscosity proposed by Stewart and Townsend (1957),

$$\eta^{(1)}=\sum_{c>0} K_c\left[\int_{\kappa}^{\infty} \mathrm{d}k\left(\frac{F(k)}{k}\right)^{1/2c}\frac{1}{k}\right]^{c} \tag{6.2}$$

obtained by retaining only the $c = 1/2$ term. They showed that the eddy viscosity (6.2) will always lead to a -7 power law for the velocity spectrum in the viscous subrange. In order to push the solution for $F(k)$ beyond the so-called viscous cutoff, k_ν, Tchen (1973) developed an alternative method for closing the hierarchy corresponding to (4.8) for $\eta^{(1)}$. He obtained a Heisenberg spectrum (1.15) with an exponential tail. The same result for $k \gg k_\lambda$ could be obtained for the thermal spectrum; however, as the larger scales of the turbulence where the anisotropic effects are evident are of principal interest in this paper, it is only noted at this point that the solution for $E(k)$ can be extended beyond k_λ.

Recently, McComb (1974) has investigated the velocity spectrum of isotropic, homogeneous and incompressible turbulence based on an equation first derived by Edwards (1964). This equation is basically of the same form as (3.1). McComb's analysis of the non-linear transfer term (corresponding to T_{ij}) leads to an equation (his equation (2.26)) which is of the same form as would be derived from the equation for $\boldsymbol{u}^{(\alpha)}(\boldsymbol{k}, t)$ using the appropriate member of the hierarchy represented by (2.9) and (2.10), where the arbitrary energy input is due to the buoyancy induced by the temperature fluctuations. In equation (3.11) there is no term which corresponds to McComb's diffusive input $H(k)$ since the rate of change of energy in a wavenumber interval beginning at $k = 0$ is considered in the present paper.

Equation (5.5), neglecting the term arising from the temperature gradient, can be compared with equation (4.10) of McComb's paper. It is seen that apart from a numerical factor, equation (5.5) is equivalent to McComb's expression for the eddy viscosity with only the first term of his expansion in the integrand retained. The viscous effect could also be included in (5.5) by incorporating the appropriate term into equation (4.5). The effect of the additional term in McComb's expansion is to change the numerical coefficients in the eddy viscosity and energy spectrum but not the variation of these quantities with wavenumber or dissipation rate. It is not completely clear that this expansion is in fact convergent, although the results in the inertial subrange would seem to indicate that this is so.

The solution of McComb in the dissipation subrange has the form of an exponential decay, obtained asymptotically for $k/k_\nu \gg 1$. For $k/k_\nu \gtrsim 1$, it might be expected that the Heisenberg spectrum would be recovered. However his spectral equation does not admit a power law solution, so that in this respect the spectrum (1.15) seems more satisfactory.

Appendix 1. Evaluation of energy generation/coupling terms

From (3.3) and (3.6) it is seen that the terms which act as energy sources in their respective equations (although B_{ij} really arises from coupling) depend upon the correlation $\langle \theta^{(0)}(\boldsymbol{k}', t) u_j^{(0)}(-\boldsymbol{k}', t)\rangle^{(0)}$. Performing a Lagrangian integration of (2.11) with the molecular term neglected, multiplying by $u_j^{(0)}(-\boldsymbol{k}', t)$, averaging and substituting the result into (3.3) gives

$$B_{ij} = \tfrac{1}{2}(M_{ij} + N_{ij}) \tag{A.1}$$

where

$$M_{ij} \equiv \alpha \hat{g}_l \chi_0 \int \mathrm{d}\boldsymbol{k}' \int \mathrm{d}\boldsymbol{k}'' \int_0^\infty \mathrm{d}t' \, \mathrm{i}k''_m \Big(\Delta_{il}(\boldsymbol{k}') \Big\langle u_j^{(0)}(-\boldsymbol{k}', t) \langle \theta^{(1)}(\boldsymbol{k}'', t') u_m^{(1)}(\boldsymbol{k}' - \boldsymbol{k}'', t')\rangle^{(1)} \Big\rangle^{(0)}$$

$$+ (i \leftrightarrow j, \boldsymbol{k}' \leftrightarrow -\boldsymbol{k}') \Big) \tag{A.2}$$

$$N_{ij} \equiv -\alpha \beta_m (g_i \eta_{mj}^{(0)} + \hat{g}_j \eta_{mi}^{(0)}) \tag{A.3}$$

and where the notation $(i \leftrightarrow j, \boldsymbol{k}' \leftrightarrow -\boldsymbol{k}')$ denotes a term identical to the immediately preceding term but with i and j interchanged and $\boldsymbol{k}'$ replaced with $-\boldsymbol{k}'$. In (A.2) and (A.3) it has been assumed that there is no memory of initial conditions so that the integration on time is over an infinite interval. In (A.3), equations (1.9) and (1.12) have been used.

The correlation $\langle \theta^{(1)} u_m^{(1)}\rangle^{(1)}$ which appears in M_{ij} can be calculated by integrating equation (2.12). In doing this, the molecular term can be neglected, as before, and also the source term, since the energy generated is input at scales larger than that of $\theta^{(1)}$. The result, when substituted into (A.2), gives

$$M_{ij} = M_{ij}(P) + M_{ij}(D)$$

where

$$M_{ij}\begin{Bmatrix} P \\ D \end{Bmatrix} = \alpha \hat{g}_l \chi_0 \int \mathrm{d}\boldsymbol{k}' \int \mathrm{d}\boldsymbol{k}'' \int_0^\infty \mathrm{d}t' \, \mathrm{i}k''_m \Big(\Delta_{il}(\boldsymbol{k}') \Big\langle u_j^{(0)}(-\boldsymbol{k}', t) \begin{Bmatrix} P_m^{(0)}(\boldsymbol{k}'', t' | \boldsymbol{k}' - \boldsymbol{k}'', t') \\ D_m^{(0)}(\boldsymbol{k}'', t' | \boldsymbol{k}' - \boldsymbol{k}'', t') \end{Bmatrix} \Big\rangle^{(0)}$$

$$+ (i \leftrightarrow j, \boldsymbol{k}' \leftrightarrow -\boldsymbol{k}') \Big) \tag{A.4}$$

with

$$P_m^{(0)}(\boldsymbol{k}'', t'|\boldsymbol{k}'-\boldsymbol{k}'', t') \equiv -\int_0^{t'} \mathrm{d}t'' \int \mathrm{d}\boldsymbol{k}\ \mathrm{i}k_n \theta^{(0)}(\boldsymbol{k}, t'') \langle u_m^{(1)}(\boldsymbol{k}'-\boldsymbol{k}'', t') u_n^{(1)}(\boldsymbol{k}''-\boldsymbol{k}, t'')\rangle^{(1)} \tag{A.5}$$

$$D_m^{(0)}(\boldsymbol{k}'', t'|\boldsymbol{k}'-\boldsymbol{k}'', t') \equiv -\int_0^{t'} \mathrm{d}t'' \int \mathrm{d}\boldsymbol{k}\ \mathrm{i}k_n \Big\langle u_m^{(1)}(\boldsymbol{k}'-\boldsymbol{k}'', t') \langle \theta^{(2)}(\boldsymbol{k}, t'') u_n^{(2)}(\boldsymbol{k}''-\boldsymbol{k}, t'')\rangle^{(2)} \Big\rangle^{(1)}. \tag{A.6}$$

Since $\theta^{(0)}(\boldsymbol{k}, t)$ is a relatively slowly varying function of time, being of rank zero, it can be removed from the integral over time in (A.5), giving

$$P_m^{(0)}(\boldsymbol{k}'', t'|\boldsymbol{k}'-\boldsymbol{k}'', t') \cong -\mathrm{i}k_n' \theta^{(0)}(\boldsymbol{k}', t') \eta_{mn}^{(1)}(\boldsymbol{k}'-\boldsymbol{k}'') \tag{A.7}$$

where (1.11) and (1.12) have also been used. Substituting (A.7) into (A.4) yields

$$M_{ij}(P) = \alpha \hat{g}_l \int \mathrm{d}\boldsymbol{k}'\ \Omega^{(1)}(\boldsymbol{k}')(\Delta_{il}(\boldsymbol{k}')\phi_j^{(0)}(\boldsymbol{k}') + \Delta_{jl}(\boldsymbol{k}')\phi_i^{(0)}(\boldsymbol{k}'))$$

where

$$\phi_i^{(\alpha)}(\boldsymbol{k}) \equiv \int_0^\infty \mathrm{d}\tau\, \mathscr{R}_i^{(\alpha)}(\boldsymbol{k}, \tau).$$

It is seen that $\phi_i^{(\alpha)}$, like the eddy viscosity, is a turbulent transport coefficient and hence shares property (1.12) with $\eta_{ij}^{(\alpha)}(\boldsymbol{k})$. Thus, since $\Omega^{(1)}(0) = 0$,

$$M_{ij}(P) = 0. \tag{A.8}$$

To find $D_m^{(0)}$, the correlation $\langle \theta^{(2)} u^{(2)} \rangle^{(2)}$ in (A.6) is formulated using the equation for $\theta^{(2)}$ which would be the third member of the hierarchy containing (2.11) and (2.12). The result is

$$\langle \theta^{(2)}(\boldsymbol{k}, t'') u_n^{(2)}(\boldsymbol{k}''-\boldsymbol{k}, t'')\rangle^{(2)} = P_n^{(1)}(\boldsymbol{k}, t''|\boldsymbol{k}''-\boldsymbol{k}, t'') + D_n^{(1)}(\boldsymbol{k}, t''|\boldsymbol{k}''-\boldsymbol{k}, t''). \tag{A.9}$$

$P_n^{(1)}$ is analogous to $P_n^{(0)}$ with all ranks increased by one and with $\theta^{(0)}$ replaced by $\theta^{(0)} + \theta^{(1)}$. Hence, by analogy with (A.7),

$$P_n^{(1)}(\boldsymbol{k}, t''|\boldsymbol{k}''-\boldsymbol{k}, t'') \cong -\mathrm{i}k_r''[\theta^{(0)}(\boldsymbol{k}'', t'') + \theta^{(1)}(\boldsymbol{k}'', t'')]\eta_{nr}^{(2)}(\boldsymbol{k}''-\boldsymbol{k}) \tag{A.10}$$

which, when substituted into (A.6) gives a zero contribution to $D_m^{(0)}$. $D_m^{(1)}$ is of the same form as $D_m^{(0)}$ except that all ranks are increased by one. Hence $D_m^{(1)}$ will contain a correlation $\langle \theta^{(3)} u^{(3)} \rangle^{(3)}$ inside the integral, which means that the equation for $\theta^{(3)}$ must be used to formulate this correlation. The non-zero part of the result will involve $\langle \theta^{(4)} u^{(4)} \rangle$ and so on. The evaluation of $D_m^{(0)}$ involves going deeper and deeper into the hierarchy. Since for small enough scales $\theta^{(\alpha)} \cong 0$ and $u^{(\alpha)} \cong 0$ (i.e. scales larger than the dissipation cut off), it follows that the ultimate contribution to $D_m^{(0)}$ found by calculating $D^{(1)}, D^{(2)} \ldots$ is zero. Therefore

$$D_m^{(0)} = 0, \qquad M_{ij}(D) = 0. \tag{A.11}$$

From (A.8) and (A.11), $M_{ij} = 0$ and therefore

$$B_{ij} = \tfrac{1}{2}N_{ij} = -\tfrac{1}{2}\alpha\beta_m(\hat{g}_i\eta_{mj}^{(0)} + \hat{g}_j\eta_{mi}^{(0)}).$$

Integrating (2.12) to obtain $\langle\theta^{(0)}u_j^{(0)}\rangle^{(0)}$, using (A.7) and following the same argument as before concerning the value of $D_m^{(0)}$ gives for C the value

$$C=\beta_j\beta_m\eta_{mj}^{(0)}.$$

Appendix 2. Evaluation of terms in equations for R_{ij} and R_j

$Q_{ij}^{(1)}$, as defined in § 4, consists of an integral over wavenumber $\boldsymbol{k}'$ with a factor $k'_m u_l^{(0)}(\boldsymbol{k}', t)$ contained in the integrand. At the higher wavenumbers where k'_m is large, $u_l^{(0)}$ is small since it is a rank zero component associated with larger scales. Hence the product is small at all wavenumbers and

$$Q_{ij}^{(1)}\cong 0. \tag{A.12}$$

A somewhat different argument leading to the same conclusion has been presented by Tchen (1973). Similar reasoning may also be applied to $\mathcal{Q}_j^{(1)}$:

$$\mathcal{Q}_j^{(1)}\cong 0. \tag{A.13}$$

Comparison of $\mathcal{L}_j^{(1)}(\boldsymbol{k}, \tau)$ with (A.6) shows that

$$\chi_1 D_j^{(0)}(\boldsymbol{k}, t|0-\boldsymbol{k}, t)=\int_0^t \mathrm{d}t''\,\mathcal{L}_j^{(1)}(\boldsymbol{k}, t''-t).$$

Utilizing (A.9) and (A.10) as well as property (1.12) gives

$$\mathcal{L}_j^{(1)}(\boldsymbol{k}, \tau)=-\Omega^{(2)}(\boldsymbol{k})\mathcal{R}_j^{(1)}(\boldsymbol{k}, \tau). \tag{A.14}$$

A similar procedure based on calculating the correlation $\langle u_i^{(1)}u_j^{(1)}\rangle^{(1)}$ instead of $\langle\theta^{(1)}u_j^{(1)}\rangle^{(1)}$ (see appendix 1) may be found in the paper of Tchen (1973), in which the expression for L_{ij} given in equation (4.3) is derived except for a numerical factor of one-half which results from a different evaluation of a complicated integration over solid angle in wavenumber space.

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